RESTRAINED BLOCK DOMINATION IN GRAPHS

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Abstract

For any (p , q) graph G , a block graph B(G) is the graph whose vertices correspond to the blocks of G and two vertices in B(G) are adjacent whenever the corresponding blocks contain a common cut vertex in G . A restrained block dominating set R ⊆ H where H = V [B(G)] and every vertex of H – R is adjacent to a vertex in R as well as another vertex in H – R . The restrained block domination number $\gamma_{rb}(G)$ is the minimum cardinality of a restrained block domination set of B(G) . We determine some upper bounds for $\gamma_{rb}(G)$ in terms of elements of G .

Key words : Block graph , restrained block domination , end edge domination , Strong split domination , restrained domination .

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Introduction :

In this paper we follow notations of [1] . All the graphs considered here are finite , simple , undirected and connected . As usual p = $|V|$ and q = $|E|$ denote the number of vertices and edges of a graph G respectively . Also n denotes number of blocks of G . The maximum degree of a vertex in G is denoted by $\Delta(G)$. A vertex v is called an end vertex if deg (v) = 1 . The Vertex covering of a graph G is a set of vertices that covers all edges of G . The vertex covering number $\alpha_0(G)$ of G is a minimum cardinality of a vertex cover in G . The greatest distance between any two vertices of a connected graph G is called the diameter of G and is denoted by diam (G) .

A spider is a tree with the property that the removal of all end paths of length two of T results in an isolated vertex called the head of a spider . Similarly an octopus is a tree with the property that removal of all end paths of length three of T results in an isolated vertex called the head of an octopus . The tentacle of an octopus is an end path of length three . We begin with some standard definitions from domination theory . Let G = (V , E) be a graph . A set D $\subseteq$ V is said to be a domination set of G if every vertex in V – D is adjacent to some vertex in D . The minimum cardinality of the set D is called the domination number of G and is denoted by $\gamma(G)$. Hedetniemi and Laskar in [4] studied connected domination . Connected dominating set D is a dominating set D whose induced sub graph < D > is connected . The connected domination number $\gamma_c(G)$ of a connected graph G is the minimum cardinality of a connected dominating set .

Degree of an edge in a graph G is deg (e) = deg(u) + deg(v) – 1 . A set F of edges in a graph G = (V , E) is called an edge dominating set of G if every edge in E – F is adjacent to at least one edge in F . The edge domination number $\gamma^I(G)$ is the minimum cardinality of an edge dominating set of G . The edge dominating set is called an end edge dominating set if all end edges belong to edge dominating set of G . The end edge domination number $\gamma_e^I(G)$ is the minimum cardinality of end edge dominating set of G . A dominating set D of a graph G is a strong split dominating set if the induced sub graph < V – D > is totally disconnected with at least two vertices . The strong split domination number $\gamma_{ss}(G)$ of a graph G is the minimum cardinality of a strong split dominating set of G .

We use the following theorems to establish our results .

Theorem A [3] : If n ≥ 2 is an integer , then $\gamma_r(K_{1,n-1}) = n$.

Theorem B [5] : If G is a graph with out isolated vertices and p ≥ 3 , then $\gamma_{ss}(G) = \alpha_0(G)$.

Theorem C [3] : If n ≥ 1 is an integer , then $\gamma_r(P_n) = n - 2\left\lfloor\frac{n-1}{3}\right\rfloor$.

Theorem D [2] : If T is a tree of order n ≥ 1 , then $\gamma_r(T) \geq \left\lceil\frac{n+2}{3}\right\rceil$.

Theorem E [4] : For any connected graph G , $\gamma_c(G) \leq p - \Delta(G)$.

Results :

Theorem 1 : For any graph G , $\gamma_{rb}(G) = 1$ if and only if

(i) G is non – separable .

(ii) G has exactly one cut vertex incident with at least three blocks .

(iii) G is a block and at least two vertices of G are incident with at least two blocks .

Proof : For necessary condition we consider the following .

For (i) , suppose G is separable with $\gamma_{rb}(G) = 1$. Then there exist at least one cut vertex . Now assume G has exactly one cut vertex . Then this condition holds for (ii) of the theorem . Hence we consider G has at least two cut vertices .

Assume G has exactly two cut vertices and each cut vertex is incident with exactly two blocks . Then clearly B(G) is $K_{1,2}$ and $\gamma_{rb}(G) > 1$, a contradiction .

Assume G has exactly two cut vertices and each cut vertex is incident with at least three blocks , which deals with condition (iii) of the statement of the theorem .

Assume G has more than two cut vertices . Then obviously $\gamma_{rb}(G) > 1$, a contradiction .

For (ii) , Suppose G has exactly one cut vertex incident with exactly two blocks and $\gamma_{rb}(G) = 1$. The block graph B(G) is an edge and clearly , $\gamma_{rb}(G) > 1$, a contradiction .

For (iii) , Suppose G is a block with $\gamma_{rb}(G) = 1$. Assume at least two vertices of G are incident with exactly one block each . Then B(G) is a star and $\gamma_{rb}(G) > 1$, a contradiction .

For sufficient condition , Suppose G holds (i) . Then clearly $\gamma_{rb}(G) = 1$.

Suppose G holds (ii) . Then B(G) is a complete graph . It follows that $\gamma_{rb}(G) = 1$.

Suppose G holds (iii) . Then B(G) has exactly one cut vertex incident with at least two complete graphs . Clearly , $\gamma_{rb}(G) = 1$.

Proposition 1 : For any (p , q) graph G with exactly one cut vertex incident with at least three blocks , then $\gamma_{rb}(G) = \gamma_{cb}(G) = \gamma_c(G) = 1$.

Proposition 2 : For any path P_p , $\gamma_{rb}(P_p) = (p-1) - 2\left\lfloor \frac{p-2}{3} \right\rfloor$ where $p \geq 2$.

Proof : Let P_p be a path with $p \geq 2$ vertices and $V[B(P_p)] = H = \{ b_1, b_2, \ldots, b_{p-1} \}$. Let $R = \{ b_1, b_2, \ldots, b_r \}$, $r < p-1$ be the sub set of H such that every vertex of H – R is adjacent to at least one vertex of R and at least one vertex of H – R . Then the set R forms a restrained block dominating set in B(G) . Now we consider the following cases .

Case 1 : If a path P_p with $p \leq 4$, then $V[B(P_p)] \leq 3$. Hence $\gamma_{rb}(P_p) = p-1$. **Case 2** : If a path P_p with $p > 4$, the vertices $b_1, b_4, b_7, \ldots, b_{p-1}$ of $H \in R$. Then the set { H – R } is a set of disjoint edges and each component of { H – R } has exactly two vertices . Suppose there are t number of edges in { H – R } . Then $2t + t + 1 \leq p-1$. So $t \leq \left\lfloor \frac{p-2}{3} \right\rfloor$. Thus $|R| = (p-1) - 2t \geq (p-1) - 2\left\lfloor \frac{p-2}{3} \right\rfloor$. The set $H - \{ b_i / 1 \leq i \leq 3\left\lfloor \frac{p-2}{3} \right\rfloor \}$, $i \equiv 2$ or $3 \pmod 3$ is a restrained block domination set of size $(p-1) - 2\left\lfloor \frac{p-2}{3} \right\rfloor$. So from above cases it follows that $\gamma_{rb}(P_p) = (p-1) - 2\left\lfloor \frac{p-2}{3} \right\rfloor$.

We obtain the relation between γ_{rb} and number of blocks n of G .

Theorem 2 : For any connected (p , q) graph G with n number of blocks , $\gamma_{rb}(G) \leq n$. Equality holds if and only if (1) G is non – separable .

(2) B(G) is a star $K_{1,n-1}$ with $n \geq 2$.

Proof : Suppose G be a graph with n number of blocks . These n blocks form vertex set in B(G) . Let $H = \{ b_1, b_2, \ldots, b_n \}$ be the vertex set in B(G) and let $R = \{ b_1, b_2, \ldots, b_r \}$, $r \leq n$ such that every vertex of H – R is adjacent to at least one vertex of R and at least one vertex of H – R . Then the set R forms a restrained block dominating set in B(G) . Since $R \subseteq H$, it follows that $|R| \leq |H|$. Hence $\gamma_{rb}(G) \leq n$.

For equality we have the following necessary condition in two cases .

Case 1 : Suppose G is separable with $\gamma_{rb}(G) = n$ where n ≥ 3 and diam (G) = 2 . Then B(G) is a complete graph and $\gamma_{rb}(G) < n$, a contradiction .

Case 2 : Suppose diam (G) > 2 and $\gamma_{rb}(G) = n$ for a graph G with n ≥ 4 blocks . Since the number of vertices in B(G) ≥ 4 , the set $R \subset H$ forms a restrained block dominating set in B(G) . Then $|R| < |H|$. It follows that $\gamma_{rb}(G) < n$. a contradiction .

Now we consider the following sufficient conditions for equality in two cases .

Case 1 : Suppose G is non – separable . Then B(G) is an isolated vertex and $\gamma_{rb}(G) = n$.

Case 2 : Suppose G is separable . Now , assume B(G) is a star . By Theorem A [3] , $\gamma_{rb}(G) = n$.

Theorem 3 : For any tree T , $\gamma_{rb}(T) \leq \gamma_e^I(T) + 1$. Equality holds for the following .

(1) T is a path P_{3k+1} , k = 1 , 2 , 3 , ……

(2) T is an octopus .

Proof : Let $A = M^I \cup E^I$ be the set of all blocks of a tree T where M^I is set of all non – end blocks and E^I is set of all end blocks of T . Let $H = M \cup E$

where H , M , E are corresponding block vertex sets of A , M^I and E^I respectively in B(T) . The set $E^I \cup I^I$ where $I^I \subseteq M^I$ forms a γ_e^I - set in T and the set $E_1 \cup I$ where $E_1 \subseteq E$ and $I \subset M$ forms γ_{rb} – set in B(T) . Clearly , $|E_1 \cup I| \leq |E^I \cup I^I| + 1$. Hence $\gamma_{rb}(T) \leq \gamma_e^I(T) + 1$. For equality we consider the following cases .

Case 1 : Suppose the tree T be a path P_{3k+1} k = 1 , 2 , 3 , ….. , then , $\gamma_e^I(P_{3k+1}) = k + 1$ and , $\gamma_{rb}(P_{3k+1}) = k + 2$. Hence the equality $\gamma_{rb}(T) = \gamma_e^I(T) + 1$ holds .

Case 2 : Suppose the tree T be an octopus . Then $\gamma_e^I(T) = |E^I \cup \{B_i\}|$ where $B_i \in M^I$ and $\gamma_{rb}(T) = |E \cup \{b_i, b_j\}|$ where $b_i, b_j \in M$. Obviously , $\gamma_{rb}(T) = \gamma_e^I(T) + 1$ holds .

Theorem 4 : For any tree $T \ncong P_3$ or P_4 , $\gamma_{rb}(T) \leq \gamma_c(T)$. Equality holds for a wounded spider with p ≥ 5 and for $K_{1,3}$.

Proof : For any tree $T \cong P_3$ or P_4 , then $\gamma_c(T) < \gamma_{rb}(T)$.

Now we consider $T \ncong P_3$ or P_4 . Let $D_c = \{ v_1 , v_2 , ……… , v_c \}$ be the connected dominating set of T . D_c contains all non – end vertices of T . $\gamma_c(T) = |D_c| = c = V -$

$|V_e|$ where V , V_e are set of all vertices and set of all end vertices respectively in T . Let $E^I = E_1^I \cup E_2^I$ be the set of all end edges of T where E_1^I is set of end edges of degree 1 and E_2^I is set of end edges of degree greater than one . The set E_1 corresponds to the set E_1^I forms set of non cut end vertices in B(T) . The set E_2 corresponds to the set E_2^I forms set of non cut non end vertices in B(T) . Let $J \subset E_2$, $E_1 \cup J$ = R forms γ_{rb}- set in B(T) . Since each vertex in B(T) contains at least one vertex of D_c , it follows that $\gamma_{rb}(T) \leq \gamma_c(T)$.

For equality , Suppose the tree T be a wounded spider . Let F = { $v_1 , v_2 , \ldots , v_f$ } be the set of all non end vertices of wounded spider . Then $\{F - \{v\}\} \cup \{v\}$ forms a γ_c - set of T and $|D_c| = |F - \{v\}| \cup |v| = \gamma_c(T)$. For wounded spider $|F - \{v\}| = |E_1|$ *and* $|v| = |J| = 1$. Hence $\gamma_{rb}(T) = \gamma_c(T)$.

Suppose T = $K_{1,3}$. Then clearly , $\gamma_{rb}(T) = \gamma_c(T) = 1$. Hence the equality .

Theorem 5 : For any tree T , $\gamma_{rb}(T) \leq \alpha_0 + 1$. Equality holds if T is isomorphic to one of the following .

(1) The tree T is a path P_3 .

(2) The tree T is an octopus with at least one tentacle .

Proof : We consider V_n = { $v_1 , v_2 , \ldots , v_j$ } be the set of vertices incident with end vertices in T and K = { $v_1 , v_2 , \ldots , v_k$ } ∈ V (T) – V_n . ∀ v_i ∈ K , $1 \leq i \leq k$, $N(v_i) \cap N(v_j) \neq \Phi$. Then { V_n } ∪ { K } gives vertex covering set . Hence $|\{V_n\} \cup \{K\}| = \alpha_0$. Now we consider E_2 = { $b_1 , b_2 , \ldots , b_i$ } , $1 \leq i \leq n$ be the set of all end vertices in B(T) and H_2 = { $b_1, b_2 , \ldots , b_j$ } , $1 \leq j \leq n$ be the set of vertices with $\deg(b_j) \geq 2$ in T . Now $\{E_2\} \cup \{H_2\} \subseteq V_n$ in T such that $V[B(T)] - \{E_2\} \cup \{H_2\}$ has at least one component . Hence $|\{E_2\} \cup \{H_2\}| = \gamma_{rb}(T)$. Clearly , $|\{E_2\} \cup \{H_2\}| \leq |\{V_n\} \cup \{K\}| + 1$ gives $\gamma_{rb}(T) \leq \alpha_0 + 1$. For equality we consider the following cases .

Case 1 : Assume $T \cong P_3$. Then clearly , $\gamma_{rb}(T) = \alpha_0 + 1$.

Case 2 : Assume T is an octopus with at least one tentacle . We consider the following sub cases .

Sub case 2.1 : Suppose T has exactly one or exactly two tentacles . Then $B(T) \cong K_{1,2}$ or $B(T) \cong P_6$. Clearly , $\gamma_{rb}(T) = \alpha_0 + 1$.

Sub case 2.2 : Suppose T has at least 3 tentacles . The vertex covering set is $\{V_n\} \cup \{K\}$ Where $V_n = \{ v_1 , v_2 , …….. , v_j \}$ is the set of all vertices incident with end vertices in T and $K = \{ v_t \}$ is a head of an octopus . $E_2 = \{ b_1 , b_2 , ….. , b_i \}$, $1 \leq i \leq n$ be the set of all end vertices in B(T) . Further $H_2 = H_2^I \cup H_3^I$, where H_2^I is set of vertices with $\deg(b_t) = 2$ and H_3^I is set of vertices with $\deg(b_l) > 2$ in B(T) . Thus $E_2 \cap H_2^I \cap H_3^I = \emptyset$. Now the set $R = E_2 \cup \{b_t\} \cup \{b_l\}$ where $b_t \in H_2^I , b_l \in H_3^I$ and $N(b_t) = b_l$ in B(T) forms γ_{rb} - set in B(T) . Since $|V_n| = |E_2|$ and $|V_t| = |b_t| = |b_l| = 1$, Then $|R| = |\{V_n\} \cup \{K\}| + 1$ gives $\gamma_{rb}(T) = \alpha_0 + 1$.

We establish the following result in terms of strong split domination .

Theorem 6 : For any tree T , $\gamma_{rb}(T) \leq \gamma_{ss}(T) + 1$ and equality holds for P_3 or octopus with at least one tentacle .

Proof : By the Theorem B [5] and Theorem 5 the result follows .

Theorem 7 : For any tree $T \neq P_3$ or P_4 if every non – end vertex of T is adjacent to at least one end vertex , then $\gamma_{rb}(T) \leq p - m$ where m is the number of end vertices in T .

Proof : Suppose $T = P_3$ or P_4 . Then $p - m < \gamma_{rb}(T)$. Hence we consider $T \neq P_3$ or P_4 . Suppose T has $p \geq 5$ vertices with $\text{diam}(T) \leq 3$. Then B(T) has exactly one cut vertex incident with two blocks or B(T) is a complete graph . Clearly the result holds .

Suppose $\text{diam}(T) > 3$ and let $V_e = \{ v_1 , v_2 , …….. , v_m \}$ be the set of all end vertices in T and $|V_e| = m$. Let $M = \{ b_1 , b_2 , ….. , b_c \}$ be the set of cut vertices in B(T) which are non – end blocks in T and $E_2 = \{ b_1 , b_2 , ….. , b_i \}$ be the set of all end vertices in B(T) . If $E_2 = \emptyset$ then $R = I$ where $I \subseteq M$ forms γ_{rb}- set in B(T) . If $E_2 \neq \emptyset$ then $R = E_2 \cup I$ forms γ_{rb}- set in B(T) . Hence in all cases $|R| \leq p - |V_e|$ gives $\gamma_{rb}(T) \leq p - m$.

Theorem 8 : For any connected (p , q) graph G , $\gamma_{rb}(G) + \gamma_r(G) \leq p + 2$.

Proof : Let $V_e = \{ v_1 , v_2 , …….. , v_m \}$ be the set of all end vertices in G . Then the set $R^I = V_e \cup I^I$ where $I^I \subset V[G] - V_e$ forms a minimum restrained dominating set in G . Let $E_2 = \{ b_1 , b_2 , ….. , b_i \}$ be the set of all end vertices in B(G) . We consider the following cases .

Case 1 : Suppose G has exactly one cut vertex then B(G) is a complete graph which gives the inequality $\gamma_{rb}(G) + \gamma_r(G) < p + 2$.

Case 2 : Suppose G has more than one cut vertex . We consider the following sub cases .

Sub case 2.1 : Assume $G \cong P_p$. Then by Theorem 1 and Theorem C [3] , $\gamma_{rb}(G) + \gamma_r(G) = p - 2\left\lfloor\frac{p-1}{3}\right\rfloor + (p-1) - 2\left\lfloor\frac{p-2}{3}\right\rfloor$

$< p + 2$

Sub case 2.2 : Assume $G \ncong P_p$. Then the set $R = E_2 \cup I_1$ where $I_1 \subset V[B(G)] - E_2$ forms a minimum restrained block domination set in B(G) . Since at least one block of G is an edge , it follows that $|R^I| \cup |R| < p + 2$ which gives $\gamma_{rb}(G) + \gamma_r(G) < p + 2$.

Theorem 9 : For any non – trivial tree T , $\gamma_{rb}(T) \leq \left\lceil\frac{p+2}{3}\right\rceil + 1$.

Proof : We prove this result by induction on the number of vertices of T . The result is true for $1 \leq p \leq 4$. Assume that this result is true for p = k vertices . Then $\gamma_{rb}(T) \leq \left\lceil\frac{k+2}{3}\right\rceil + 1$.

Suppose e is added to any non end vertex of T . Then this e together with its neighborhood N(e) form a complete block in B(T) and $\gamma_{rb}(T) \leq \left\lceil\frac{(k+1)+2}{3}\right\rceil + 1$ which gives the required result . Suppose e is added to any end vertex of T . Then $e \in V[B(T)]$ as an end vertex . Clearly , $\gamma_{rb}(T) \leq \left\lceil\frac{(k+1)+2}{3}\right\rceil + 1$ holds . Hence $\gamma_{rb}(T) \leq \left\lceil\frac{p+2}{3}\right\rceil + 1$.

Theorem 10 : For any tree T , $\gamma_{rb}(T) \leq \gamma_r(T) + 1$. Equality holds if T is isomorphic to one of the following .

(1) The tree T is an octopus .

(2) The tree T is a path P_{3k+1}, k = 1,2,3,…..

Proof : From Theorem 9 , we have $\gamma_{rb}(T) \leq \left\lceil\frac{p+2}{3}\right\rceil + 1$ …………. (1)

Also by Theorem D [2] , $\left\lceil\frac{p+2}{3}\right\rceil \leq \gamma_r(T)$.

Hence $\left\lceil\frac{p+2}{3}\right\rceil + 1 \leq \gamma_r(T) + 1$ ……….. (2)

From (1) and (2) , $\gamma_{rb}(T) \leq \gamma_r(T) + 1$.

For equality we consider the following cases .

Case 1 : Suppose a tree T is an octopus . Let $V = V_e \cup V_1 \cup V_2$ be the vertex set of T where V_e is set of all end vertices , V_1 is set of all non – end vertices with deg (v_l) = 2 and V_2 is the set of non – end vertices with deg (v_s) > 2 . Then $R^I = V_e \cup V_2$ forms a γ_r – set in T and

$|R^I| = |V_e| + |V_2| = |V_e| + 1$ . ... (1)

Let A = { B_1 , B_2 , ………. , B_n } be the set of blocks of T . The set $A = {M_1}^I \cup {M_2}^I \cup E^I$ where ${M_1}^I$ is set of non – end blocks of degree 2 , ${M_2}^I$ is set of non – end blocks of degree greater than 2 and E^I is set of end blocks of degree 1 in T . The corresponding block vertex sets of ${M_1}^I$, ${M_2}^I$ *and* E^I are M_1 , M_2 and E respectively in B(T) . The set $R = E \cup \{b_i\} \cup \{b_k\}$ for some b_i and b_k such that $b_i \in M_1 , b_k \in M_2$ form a γ_{rb} - set in B(T) . Hence $|R| = |E| + |b_j| + |b_k| = |E| + 1 + 1$. …………… (2)

From equations (1) and (2) the equality holds .

Case 2 : Suppose a tree $T \cong P_{3k+1}$, k = 1, 2 , 3 , …… . The block graph $B(T) \cong P_{3k}$, k = 1, 2 , 3 , …… . The minimum restrained domination number $\gamma_r(P_{3k+1})$ = k + 1 and $\gamma_{rb}(P_{3k+1})$ = k + 2 . Hence $\gamma_{rb}(T) = \gamma_r(T) + 1$.

Theorem 11 : If v be an end vertex of a connected block graph B(G) , then v is in every γ_{rb} - set of B(G) .

Proof : Suppose R be a minimum restrained block dominating set of B(G) . Assume ∃ an end vertex v ∈ V [B(G)] – R . Then v should be adjacent to at least one vertex of V [B(G)] – R and at least one vertex of R which shows that $|N(v)| > 1$ *or* $\Delta(G) > 1$, which is a contradiction . Hence $v \notin$ V [B(G)] – R and v is in every γ_{rb} - set of B(G) .

Theorem 12 : For any tree T , $\gamma_{rb}(T) \leq p - \Delta(T) + 1$.

Proof : By Theorem 4 , $\gamma_{rb}(T) \leq \gamma_c(T)$ -------------- (1) and by Theorem E [4] , $\gamma_c(G) \leq p - \Delta(G)$. Then $\gamma_c(G) + 1 \leq p - \Delta(G) + 1$.

So $\gamma_c(T) + 1 \leq p - \Delta(T) + 1$ -------------------- (2) . Combining equations (1) and (2) gives $\gamma_{rb}(T) \leq \gamma_c(T) \leq \gamma_c(T) + 1 \leq p - \Delta(T) + 1$. Hence $\gamma_{rb}(T) \leq p - \Delta(T) + 1$.

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